Extended formulations for higher-order spanning tree polytopes

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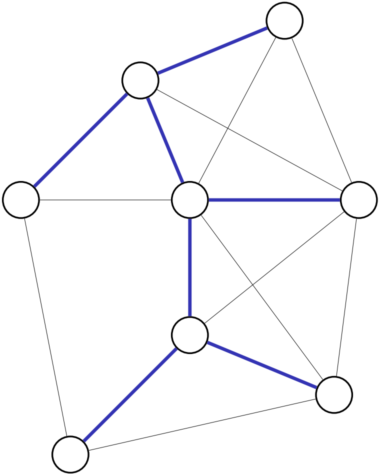
Outline

- Higher-order spanning tree polytopes
- Motivation
- Extended formulations
- Properties

The spanning tree polytope

$$P_{ST} := \text{conv}\left\{\chi(T) \in \{0,1\}^E \mid T \text{ is a spanning tree of } G = (V, E)\right\}$$

$$\chi(T)_e := \begin{cases} 1 & \text{if } e \in T \\ 0 & \text{otherwise} \end{cases}$$



Higher-order spanning tree polytopes

For a set of monomials $\mathcal{M} \subseteq 2^E$ we define

$$P(\mathcal{M}) := \text{conv}\left\{(x, y) \in \{0,1\}^{|E|+|\mathcal{M}|} \;\middle|\; \begin{array}{ll} x = \chi(T) & T \text{ is spanning tree,} \\ y_M = \Pi_{e \in M} x_e & \text{for all } M \in \mathcal{M} \end{array} \right\}$$

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examples

- $\mathcal{M} = \{M \in 2^E \mid |M| = 2\} \quad \rightarrow P(\mathcal{M}) = P_{QST}$
- $\mathcal{M} = \{M\},\ |M| = 2$ [Buchheim and Klein 2014] [Fischer and Fischer 2013]
- $\mathcal{M} = \{M_1, \ldots, M_k\},\ M_1 \subset M_2 \subset \cdots \subset M_k$ [Fischer et al. 2017]
- $\mathcal{M} = \{M \in 2^F \mid F \subset E\}$ [Fischer et al. 2017]

One quadratic term technique

Buchheim and Klein 2014

$$C(M) := \text{conv}\left\{(x,y) \in \{0,1\}^{|E|+|\mathcal{M}|} \mid (x, y_M) \in P(\{M\})\right\}$$

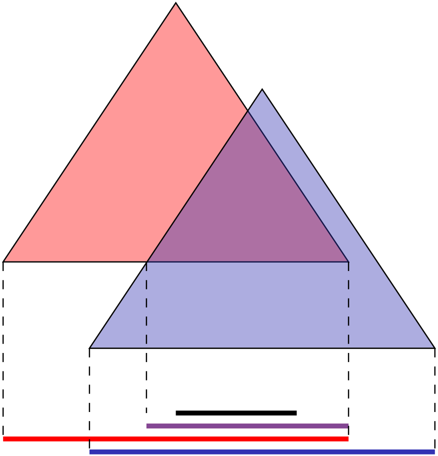
for $M \in \mathcal{M}$

$\Rightarrow$ Relaxation: $$P_{QST} \subseteq \bigcap_{M \in 2^E,\ |M|=2} C(M)$$

$\uparrow$

$$\boxed{\mathcal{M} = \{M \in 2^E \mid |M| = 2\}}$$

Combining extended formulations and the one quadratic term technique

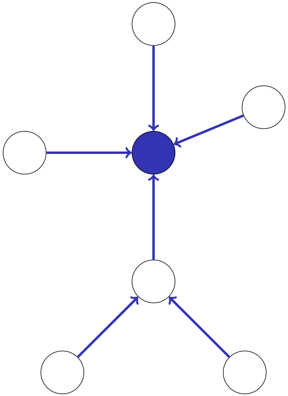


$$\text{proj}(Q_1 \cap Q_2) \subseteq \text{proj}(Q_1) \cap \text{proj}(Q_2)$$

Extended formulation for the linear spanning tree polytope

Martin 1991

$$z^u_{v,w} + z^u_{w,v} = x_{\{v,w\}}$$
$$z^u\left(\delta^{out}(v)\right) = 1$$
$$z^u\left(\delta^{out}(u)\right) = 0$$
$$z^u_{v,w} \geq 0$$



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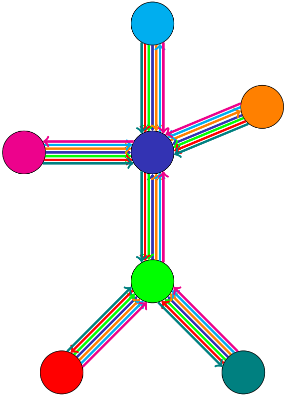
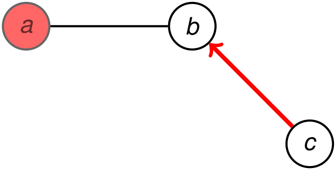


Adjacent quadratic product

$$\boxed{y \leq z^{a}_{c,b}} \qquad \leq x_{\{b,c\}}$$

$$y \leq x_{\{a,b\}}$$

$$y \geq x_{\{a,b\}} + x_{\{b,c\}} - 1$$



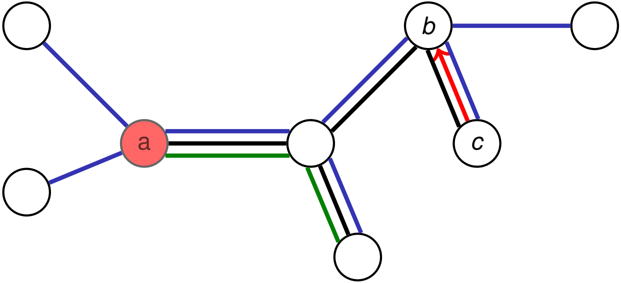
Theorem

Extended formulation of $P(\mathcal{M})$ where $\mathcal{M} = \{\{e_1, e_2\}\}$, and e_1 and e_2 share a common node

Nested trees

$y_{T_3} \leq y_{T_2} \leq z^a_{c,b}$

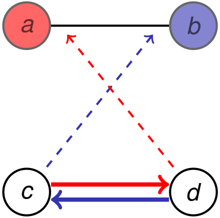
...



Theorem

Extended formulation of $P(\mathcal{M})$ *where* $\mathcal{M} = \{T_1, T_2, \ldots, T_k\}$ *and* $T_1 \subset T_2 \subset \cdots \subset T_k$ *are trees*

Non-adjacent quadratic product



$$\boxed{y \leq z^a_{c,d} + z^b_{d,c}}$$

$$\boxed{y \leq z^a_{d,c} + z^b_{c,d}}$$

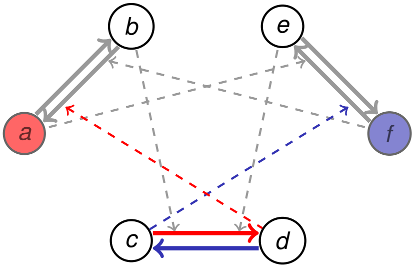
$$y \leq x_{a,b}$$

$$y \geq x_{\{a,b\}} + x_{\{c,d\}} - 1$$

Theorem

Extended formulation of $P(\mathcal{M})$ *where* $\mathcal{M} = \{\{e_1, e_2\}\}$

non-adjacent cubic product



$$2y \leq z^{a}_{c,d} + z^{f}_{d,c} + z^{c}_{a,b} + z^{e}_{b,a} + z^{b}_{e,f} + z^{d}_{f,e}$$

...

Theorem

Extended formulations of $P(\mathcal{M})$ where

- $\mathcal{M} = \{\{e_1, e_2, e_3\}\}$
- $\mathcal{M} = \{\{e_1, e_2\}, \{e_1, e_2, e_3\}\}$

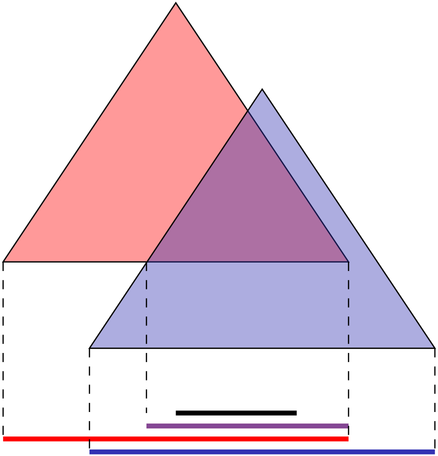
Proof idea

Validity as discussed before.

To show that the formulation is sufficient use complete description of $P(\mathcal{M})$ provided by

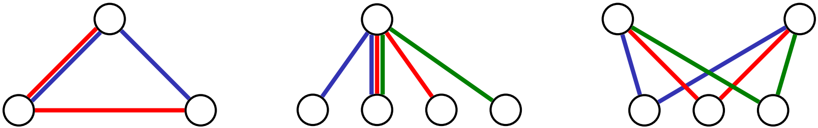
- Christoph Buchheim and Laura Klein (2014) for one quadratic monomial and
- Anja Fischer, Frank Fischer and S. Thomas McCormick (2017) for nested monomials

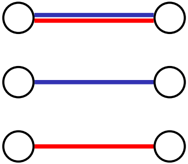
Improvement of the one quadratic term technique



$$\text{proj}(Q_1 \cap Q_2) \subseteq \text{proj}(Q_1) \cap \text{proj}(Q_2)$$

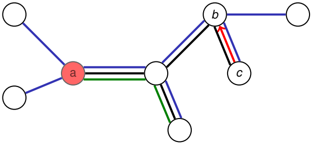
Combination of several quadratic monomials

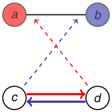


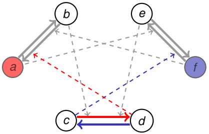


- New facets of $P_{AQST} := P(\mathcal{M})$ for $\mathcal{M} = \{\{e, f\} \in 2^E |\ e, f \text{ are adjacent}\}$

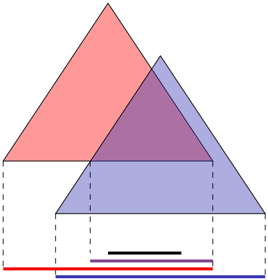
Conclusion



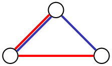


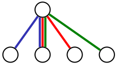


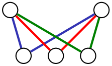
We used additional structural information of an extended formulation and obtained:



- Extended formulations for higher-order spanning tree polytopes namely for
  - one quadratic monomial
  - monomials that are nested trees
  - nested monomials up to degree 3
- a stronger relaxation of P_{QST}
- new facets of P_{AQST}







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